Simple Waves in Saturated Porous Media*
(I. The Isothermal Case)

A. KRYLOV**, S. SOREK***, A. LEVY****
and G. BEN-DOR****

In a previous paper (Bear et al. Fluid Dynamics Research, Vol. 9, pp. 155-164) a mathematical model was developed in order to analyze an abrupt pressure impact applied to a compressible fluid flowing through saturated, porous media under isothermal conditions. It was shown that during a certain time period following the onset of pressure change, the macroscopic fluid momentum balance equation conforms to a wave form. This paper presents a one-dimensional simple analytical solution to this wave equation. The wave equation is transformed to Euler's equation describing the motion of a "new" fluid with characteristics related to the former one.

Key Words: Saturated Porous Media, Homogeneous, Isotropic, Isothermal, Euler's Equation, Abrupt Pressure Change

1. Introduction

Our aim is to analyze the spatial and temporal changes due to pressure imposed on a fluid phase that occupies the entire void space of a porous medium. To address this, Bear and Sorek[1] studied the non dimensional forms of the macroscopic fluid mass and momentum balance equations. They showed, that following the onset of the pressure change, there was a sequence of four time periods for which the dominating mass and momentum balance equations were different. During the second time interval, the fluid momentum balance equation (i.e., the equation of motion of the fluid) conformed to a wave equation. Papers which emerged since dealt with a deformable porous media under non-isothermal conditions [Bear et al.[2]; Sorek et al.[3]] and with abrupt simultaneous changes in the pressure and temperature of the fluid [Levy et al.[4]].

Biot's[5] work on compression wave propagation was probably the first one to employ the fundamental concepts of porous media mechanics. A large number of papers have appeared in the literature following his pioneering work. Among the recent ones are those by Smeulders et al.[6], Degrande and De Roeck[7] and Nigmatulin and Gubaidullin[8]. These refer to microscopic representations of the balance equations of the phases with the framework of the theory of mixtures. Most refer to linear waves which are generated when the dissipation terms of the momentum equation dominate. As an example, Attenborough[9] presented a theory dealing with the motion of sound waves through an ideal saturated porous matrix with parallel cylindrical pores. Using microscopic physical parameters, he extended this to account for a random distribution of pores. An extensive literature survey

* Received 11th January, 1995.

** Visiting Professor from the Institute for Earth Physics of the Russian Academy of Science, Moscow, Russia

*** Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer Sheva, Israel, also affiliated with the Water Resource Research Center, J. Blaustein Desert Research Institute, Sede Boker Campus 84990, Israel

**** Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer Sheva, Israel

of similar approaches was given by Corapcioglu[10]. The model developed in the course of this study is different from those mentioned earlier in its approach and hence yields different terms in the governing equation. It is novel in establishing a macroscopic theoretical basis for the motion of waves in multi-phase deformable porous media. The porous medium was modeled as a continuum composed of interacting compressible solid and multiple fluid phases (e.g., liquid and gas). Macroscopic physical laws expressing the mass and momentum balances for the fluids phases and the solid matrix were formulated on the basis of Representative Elementary Volume (REV) concepts. The generation of motion resulted in a situation in which the inertial terms in the momentum equation were dominant. In turn, a nonlinear wave equation was obtained. The present model, at the macroscopic level, deals with the second time interval at which the motion equations of the fluid and solid phases conform to wave equations.

2. The Conceptual Model

The conceptual model may be summarized by the following set of simplifying assumptions:

[A. 1] The fluid is Newtonian and compressible.

[A. 2] The solid phase is incompressible.

[A. 3] The solid matrix is elastic and assumed to undergo only small deformations.

[A. 4] The dispersive flux of momentum is much smaller than its advective one and may, therefore, be neglected.

[A. 5] The dispersive and diffusive mass fluxes of the fluid, and the dispersive flux of the solid, are much smaller than the corresponding advective ones and may, therefore, be neglected.

[A. 6] The microscopic solid-fluid interfaces are material surfaces (i.e., the fluid cannot penetrate the solid).

[A. 7] The stress-strain relationship for the solid, at the microscopic level, and for the solid matrix, at the macroscopic one, have the same form.

3. Balance Equations

Following Bear and Sorek[1] with [A. 1], [A. 4], [A. 5] and [A. 6] we find that during the second time interval following an abrupt pressure change, the dominant one-dimensional mass and momentum balance equations reduce to the form:

Fluid mass balance equation.

$$\frac{\partial}{\partial t}(\phi\rho_f)+\frac{\partial}{\partial x}(\phi\rho_f u_f)=0, \qquad (1)$$

in which ϕ denotes the porosity, u_f the velocity of the fluid and ρ_f its density, which is assumed to be a function of pressure, p. Hence,

Fluid state function.

$$\rho_f=\rho_f(p). \qquad (2)$$

Fluid momentum balance equation

This motion equation conforms to a wave equation. In the 1-D horizontal direction, this gives

$$\frac{\partial u_f}{\partial t}+u_f\frac{\partial u_f}{\partial x}+\frac{1}{\rho_f}\frac{\partial p}{\partial x}T^*=0, \qquad (3)$$

in which T^* denotes the tortuosity coefficient.

Following Bear et al.[2] and Sorek et al.[3] and taking into account assumptions [A. 2], [A. 4] and [A. 5] results in

Solid mass balance equation.

$$\frac{\partial(1-\phi)}{\partial t}=-\frac{\partial}{\partial x}[(1-\phi)u_s], \qquad (4)$$

in which u_s denotes the solid phase velocity.

In view of assumptions [A. 2] and [A. 3], and following Levy et al.[4] for an elastic solid matrix, we write

Solid momentum balance equation.

$$\frac{\partial u_s}{\partial t}+u_s\frac{\partial u_s}{\partial x}-\frac{\lambda_s}{\rho_s(1-\phi)}\frac{\partial^2 w}{\partial x^2}+\frac{T^*}{\rho_s}\frac{\partial p}{\partial x}=0, \qquad (5)$$

where λ_s denotes the Lamé constant of an elastic solid; ρ_s, which according to assumption [A. 2] is constant, denotes its material density; and w denotes the displacement of the solid skeleton.

4. Approximate Forms of Solid Mass and Momentum Balance Equations

We consider a traveling wave propagating, say, in the positive direction of the x-axis. In such a case all the quantities are functions of (x-ct), where c denotes the linear propagation speed of the wave. Thus, the porosity may be represented by the real part of a complex expression

$$\phi=Re\{A\exp[-i(\omega t-kx)]\}, \qquad (6)$$

where ω denotes the wave frequency and k denotes a displacement per unit length. Let us now assume that

$$\left|\frac{\partial(1-\phi)}{\partial t}\right|\gg\left|u_s\frac{\partial(1-\phi)}{\partial x}\right|. \qquad (7)$$

In view of Eqs.(6) and (7) we obtain

$$c=\left|\frac{\omega}{k}\right|\gg|u_s|. \qquad (8)$$

Furthermore, (7) can be viewed in terms of Struhal number. Let us define the Struhal number, St^e, associated with an intensive quantity, e as

$$St^e\equiv\frac{L_c^e}{t_c^e u_c^\alpha}, \qquad (9)$$

where L_c^e and t_c^e denote the characteristic space increment of e and its time step, respectively, and u_c^α denotes the velocity of the α-phase ($\alpha\equiv$fluid or solid). The characteristic intensive quantities, $(\ )_c^e$, are chosen in such a way that the non-dimensional flux terms embedded in the balance equations will be of unit order. By virtue of Eq.(7) and letting $\alpha\equiv s$ and $e\equiv\phi$, we obtain

$$St^{\phi}\gg 1. \tag{10}$$

In view of Eqs.(7) or (8) and (10), we rewrite Eq. (4) in the form

$$\frac{\partial\phi}{\partial t}=(1-\phi)\frac{\partial u_s}{\partial x}. \tag{11}$$

We define

$$u_s=\frac{D_s w}{Dt}, \tag{12}$$

in which the material derivative, $\frac{D_a}{D_t}$, is defined by

$$\frac{D_a}{D_t}\equiv\frac{\partial}{\partial t}+u_a\frac{\partial}{\partial x}. \tag{13}$$

Based on assumption (8) or an assumption similar to assumption (10) when $e\equiv w$ $\left(\text{i.e., } St^w\equiv\frac{L_c^w}{t_c^w u_c^s}\gg 1\right)$ we may rewrite Eq.(12) in the form

$$u_s\cong\frac{\partial w}{\partial t}. \tag{14}$$

The solid skeleton strain, ε, is related to, w, by the compatibility equation, namely

$$\varepsilon=\frac{\partial w}{\partial x}. \tag{15}$$

Hence, in view of Eqs.(11), (14) and (15) we may write the mass balance equation of the solid in the form

$$\frac{1}{1-\phi}\frac{\partial\phi}{\partial t}=\frac{\partial u_s}{\partial x}=\frac{\partial\varepsilon}{\partial t}. \tag{16}$$

As in Eq.(6), we write the temporal and spatial changes of Eq.(16) in a complex representation as

$$-i\omega\varepsilon=iku_s \tag{17}$$

or

$$\left|\frac{\omega}{k}\right||\varepsilon|=|u_s|. \tag{18}$$

By virtue of Eqs.(8) and (18) we conclude that

$$|\varepsilon|\ll 1. \tag{19}$$

Integrating Eq.(16) and in view of Eq.(19) we obtain

$$\frac{1-\phi}{1-\phi_0}=\exp(-\varepsilon)\cong 1-\varepsilon, \tag{20}$$

in which we assume that $\varepsilon_{(x,0)}=0$ and where $\phi_0(=\phi_{(x,0)})$ denotes an initial porosity.

Substituting Eq.(14) into Eq.(5) yields the following equation of motion for the solid matrix

$$\frac{\partial^2 w}{\partial t^2}-c_s^2\frac{\partial^2 w}{\partial x^2}+\frac{T^*}{\rho_s}\frac{\partial p}{\partial x}=0, \tag{21}$$

where c_s, the propagation speed of the displacement wave in the solid phase, is given by

$$c_s^2\equiv\frac{\lambda_s}{\rho_s(1-\phi)}. \tag{22}$$

We now consider the assumption

$$St^w M_s\ll 1, \tag{23}$$

where $M_s\left(\equiv\frac{u_s}{c_s}\right)$ denotes the solid Mach number.

Based on the order of magnitude of St^w [see the text following Eq.(13)] and Eq.(23), we may assume that

$$M_s\ll 1. \tag{24}$$

This implies that the first two terms in Eq.(5) are negligibly small compared to the others, i.e., the inertia of the solid is negligible. Therefore Eq.(5) can be rewritten as:

$$-\frac{\lambda_s}{1-\phi}\frac{\partial\varepsilon}{\partial x}+T^*\frac{\partial p}{\partial x}=0. \tag{25}$$

Substituting Eq.(20) into Eq.(25) and in view of Eq.(19), we solve Eq.(25) for ε, and obtain

$$\varepsilon=\frac{1-\phi_0}{\lambda_s}T^*(p-p_0), \tag{26}$$

where $\varepsilon=0$ at $p=p_0$.

In view of Eq.(20), we may write Eq.(26) in terms of the porosity, namely

$$\phi=\phi_0+\frac{(1-\phi_0)^2}{\lambda_s}T^*(p-p_0). \tag{27}$$

Using the compatibility equation expressed in Eq.(15) and in view of Eq.(26) we write

$$w=\frac{1-\phi_0}{\lambda_s}T^*\int_{-\infty}^{x}(p-p_0)dx, \tag{28}$$

where we have assumed that $w_{(-\infty,t)}=0$.

Combining Eqs.(14) and (28) we write

$$u_s=\frac{1-\phi_0}{\lambda_s}T^*\int_{-\infty}^{x}\frac{\partial}{\partial t}(p-p_0)dx. \tag{29}$$

We now aim at rewriting the mass and momentum balance equations of the fluid in a form equivalent to Euler's equation describing the propagation of simple fluid waves.

5. Approximate Forms of Fluid Mass and Momentum Balance Equations

We note that in order to rewrite the fluid mass and momentum balance equations to conform to a simple hydrodynamic wave equation, we have to convert the pressure-dependent variables. Let us, thus define a "new" fluid characteristic, $(\ )_n$, by the following

$$\rho_{n(p_n)}=\phi\rho_{f(p_{(p_n)})}, \tag{30}$$

where in view of Eqs.(3), (30) and Euler's equation, we define p_n, in order to find the expression $p_{(p_n)}$, by

$$\frac{T^*\phi_{(p)}}{\rho_{n(p)}}\frac{\partial p}{\partial x}=\frac{1}{\rho_{n(p)}}\frac{\partial p_{n(p)}}{\partial x}. \tag{31}$$

Substituting Eq.(27) into Eq.(31), gives

$$T^*\left[\phi_0+\frac{(1-\phi_0)^2}{\lambda_s}T^*(p-p_0)\right]=\frac{dp_n}{dp}. \tag{32}$$

The solution of Eq.(32), assuming $p_{n(x,0)}=0$, yields

$$p_{n(p)}=\left[\phi_0+\frac{(1-\phi_0)^2T^*}{2\lambda_s}(p-p_0)\right](p-p_0)T^*. \tag{33}$$

In view of Eqs.(30) and (31) we rewrite Eqs.(1) and (3), respectively

$$\frac{D_f\rho_n}{Dt}+\rho_n\frac{\partial u_f}{\partial x}=0 \tag{34}$$

and

$$\frac{D_f u_f}{Dt}+\frac{1}{\rho_n}\frac{\partial p_n}{\partial x}=0, \tag{35}$$

where the hydrodynamic time derivative is defined by

$$\frac{D_f}{Dt}\equiv\frac{\partial}{\partial t}+u_f\frac{\partial}{\partial x}.$$

In view of Eqs. (27) and (30) we rewrite Eq. (2), the fluid state function

$$\rho_n=\rho_{n(p_n)}=\left\{\phi_0+\frac{(1-\phi_0)^2}{\lambda_s}T^*[p_{(p_n)}-p_0]\right\}\rho_{f(p_{(p_n)})}. \tag{36}$$

Equations (34) to (36) compose the complete system of equations describing the motion of the "new" fluid.

6. Simple (Riemann) Waves

Actually, Eqs. (34) to (36) describe the equivalent to the traveling of a 1-D simple wave. Thus, following Landau and Lifshitz[11], and using the requirement that velocity is a function of density, we find

$$x=t[u_{n(p_n)}\pm c_{n(p_n)}]+f_{(p_n)}, \tag{37}$$

in which the initial displacement, i.e., $x(0)$ is $f_{(p_n)}$, and u_n, which denotes the velocity of the "new" fluid is given by

$$u_{n(p_n)}=\pm\int_0^{p_n}\frac{dP_n}{\rho_n c_n}. \tag{38}$$

The sound speed, c_n, of this fluid is given by

$$c_n=\pm\left(\frac{d\rho_n}{dp_n}\right)^{-1/2}. \tag{39}$$

Since the value of, ρ_n, uniquely determines the value of, u_n, we can use either Eq. (34) or Eq. (35) and write the pressure wave equation of the "new" fluid in the form

$$\frac{\partial p_n}{\partial t}+(u_n\pm c_n)\frac{\partial p_n}{\partial x}=0. \tag{40}$$

7. Characteristics of the "New" Fluid

We aim to determine the relations between the parameters associated with the fluid in comparison to the "new". In view of Eqs. (22), (27), (30) and (31), we expand Eq. (39) to yield

$$c_n^2=\frac{\phi}{\rho_{s_0}}T^* =\left[1+\frac{(1-\phi_0)^2}{\beta\lambda_s\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}T^*\right]^{-1}T^*c_f^2, \tag{41}$$

where s_0 which denotes the specific storativity of the porous medium is given by

$$s_0\equiv\phi\beta+(1-\phi)\alpha, \tag{42}$$

in which

$$\beta\equiv\frac{1}{\rho_f}\frac{d\rho_f}{dp}=\frac{1}{\rho_f c_f^2} \tag{43}$$

denotes the compressibility of the fluid, and

$$\alpha\equiv\frac{1}{1-\phi}\frac{d\phi}{dp} \tag{44}$$

denotes the compressibility of the porous medium as a whole. The compressibility, β_n, and velocity, u_n, of the "new" fluid are obtained from Eqs. (27), (30) and (42) to (44). They are respectively,

$$\beta_n\equiv\frac{1}{\rho_n}\frac{d\rho_n}{dp_n}=\frac{s_0}{\phi^2}T^{*-1} =\frac{\beta}{\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}T^{*-1}\left[1+\frac{(1-\phi_0)^2}{\beta\lambda_s\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}T^*\right] \tag{45}$$

$$\frac{du_n}{dp_n}=\left(\frac{\beta_n}{\rho_n}\right)^{1/2} =\frac{T^{*-1/2}}{\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}\left[1+\frac{(1-\phi_0)^2}{\beta\lambda_s\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}T^*\right]^{1/2}. \tag{46}$$

Next we investigate the order of magnitude of the "new" fluid parameters. Let us consider solid porous materials such as SiC or Al_2O_3 for which $\lambda_s\cong 400$ GPa. Considering Eq. (27), we note that $\Delta p\ (\equiv p-p_0)$ is many orders of magnitude smaller than λ_s. This is exemplified in the case of a head-on reflection of shock waves where the pressure change is $14P_0$ for a typical Mach number of $M_f=2$ [for details see Liepmann and Roshko[12]]. Hence, for atmospheric pressure ($P_0=100$ kPa) the second term on the R.H.S. of Eq. (27) is negligible. Thus we may conclude that the porosity, ϕ, remains approximately constant, i.e.,

$$\phi=\phi_0+\xi(1-\phi_0)^2T^*\cong\phi_0, \tag{47.1}$$

where ξ is defined by

$$\xi\equiv\frac{p-p_0}{\lambda_s}\ll 1. \tag{47.2}$$

Following Eq. (47.2), we may approximate Eq. (33) to yield a linear relation with respect to ϕ_0 and T^*, namely

$$p_n=(p-p_0)\phi_0T^*+\xi(1-\phi_0)^2T^*(p-p_0) \cong(p-p_0)\phi_0T^*. \tag{48}$$

In obtaining an order of magnitude for c_n in Eq. (41), we note that (for example in the case of $\lambda_s=400$ GPa, $\rho_s=2000\ \mathrm{kg/m^3}$ and $c_s=7000$ m/sec) we always have

$$\frac{\lambda_s}{\rho_s C_s^2}>1, \tag{49}$$

hence Eq. (41) can be approximated by

$$c_n^2=\left[1+\delta\frac{(1-\phi_0^2T^*)}{\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}\right]T^*c_f^2\cong T^*c_f^2, \tag{50.1}$$

where δ is defined by

$$\delta\equiv\frac{1}{\beta\lambda_s}\ll 1. \tag{50.2}$$

Figure 1 depicts the relations $c_{n_1}^2\equiv T^*c_f^2$ and $c_{n_2}^2\equiv\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)^{-1}[(1-\phi_0)T^*]^2c_f^2$ which, as expected, are dierent from each other by many orders of magnitude. Note, however, that c_{n_1} and c_{n_2} do not indicate the existence of two different waves; they simply express two expansion modes of $c_n^2\ (\equiv c_{n_1}^2+c_{n_2}^2)$. We note that c_{n_1} is similar to the "slow wave velocity" given by Johnson and Plona[13].

In view of Eq. (50.2), we may rewrite Eq. (45) as

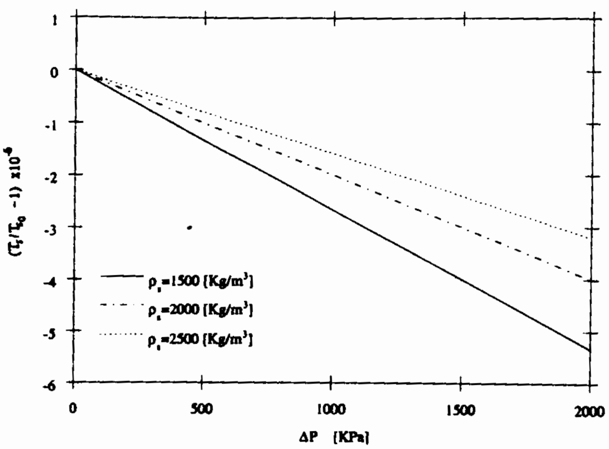

Fig. 1 Ratio of the solid temperature vs pressure drop for varioud solid densities ($T^*=0.8$, $c_c'=1000$ J/kg/K, $\lambda_s=400$ [GPa], $\eta=2$ MPa/K)

$$\beta_n=\frac{1}{T^*\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}\beta+\left(\frac{1-\phi_0}{1-\frac{\lambda_s}{\rho_s C_s^2}}\right)^2\delta\beta$$
$$\cong\frac{1}{T^*\left(1-\frac{\lambda_s}{\rho_s C_s^2}\right)}\beta \tag{51}$$

from which it is clearly seen that β_n depends linearly on β. Note also that by following Eq.(49) and since $\beta>0$, Eq.(51) implies that $\beta_n<0$.

8. Conclusion

A simple one-dimensional analytical model was developed to represent the wave which is generated in a saturated deformable porous media following the onset of a pressure impact. It was shown that the fluid mass and momentum balance equations conform to Riemann's problem associated with a "new" fluid pressure and density. The expressions for the sound speed. compressibility and fluid velocity of this "new" fluid, were determined in relation to the same properties of the original fluid. Approximate forms of these expressions were proven to yield linear functions between properties of the same nature of both fluids.

References

(1) Bear, J. and Sorek, S., Evolution of Governing Mass and Momentum Balances Following an Abrupt Pressure Impact in Porous Medium, Transport in Porous Media, Vol. 5(1990), p. 169.
(2) Bear, J., Sorek, S., Ben-Dor, G. and Mazor, G., Displacement Waves in Saturated Thermoelastic Porous Media. I. Basic Equations, Fluid Dynamic Research, Vol. 9(1992), p. 155.
(3) Sorek, S., Bear, J., Ben-Dor, G. and Mazor, G., Shock Waves in Saturated Thermoelastic Porous Media, Transport in Porous Media, Vol. 9(1992), p. 3.
(4) Levy, A., Sorek, S., Ben-Dor, G. and Bear, J., Evolution of the Balance Equations in Saturated Thermoelastic Porous Media Following Abrupt Simultaneous Changes in Pressure and Temperature, Transport in Porous Media, Vol. 21(1996), p. 241.
(5) Biot, M.A., Theory of Propagation of Elastic Waves in Fluid-Saturated Porous Solid, J. Acous. Soc. An., Vol. 28(1956), p. 168.
(6) Smeulders, D.M.J., De La Rosette, S.P.M. and Van Dongen, M.E.H., Waves in Partially Saturated Porous Media, Transport in Porous Media, Vol. 9 (1992), p. 25.
(7) Degrande, G. and De Roeck, G., FFT-Based Spectral Analysis Methodology for One-Dimensional Wave Propagation in Poroelastic Media, Transport in Porous Media, Vol. 9(1992), p. 85.
(8) Nigmatulin, R. I. and Gubaidullin, A.A., Linear Waves in Saturated Porous Media, Transport in Porous Media, Vol. 9(1992), p. 135.
(9) Attenborough, K., Acoustical Characteristics of Porous Materials, Physics Reports, Vol. 82, No. 3 (1982), p. 179.
(10) Corapcioglu, M.Y.,"Wave Propagation in Porous Media - a Review" in Bear, J. and Corapcioglu, M.Y. (eds.), Transport Processes in Porous Media, Kluwer Acad, Pub., Dordrect, (1991), p. 373.
(11) Landau, L.D. and Lifshitz, E.M., Fluid Mechanics, 2nd Edition, (1987), p. 378, Pergamon Press.
(12) Liepmann, H.W. and Roshko, A., Elements of Gasdynamics, (1957), John Wiley and Sons, New York.
(13) Johnson, D.L. and Plona, T.J., Acoustic Slow Waves and the Consolidation Transition, J. Acoust. Soc. Am., Vol. 72, No. 2(1982), p. 556.